European Water 57: 435-441, 2017.

A two-dimensional analytical model for tide-induced groundwater fluctuation in leaky aquifers

M.-H. Chuang[1] and H.-D. Yeh[2*]
[1] *Department of Urban Planning and Disaster Management, Ming-Chuan University, Taoyuan, Taiwan*
[2] *Institute of Environmental Engineering, National Chiao Tung University, Hsinchu, Taiwan*
[*] *e-mail: hdyeh@mail.nctu.edu.tw*

Abstract: Tide-induced groundwater head fluctuations in coastal leaky aquifer systems are complicated and difficult to quantify. Yet, the prediction of groundwater fluctuations may be a crucial work in dealing with the problems of groundwater resources management or contamination remediation in coastal aquifers. This study develops a two-dimensional analytical model for describing the groundwater flow in an *L*-shaped leaky aquifer bounded by a river and the ocean with an arbitrary included angle and subject to the effect of tidal fluctuations. The solution of the model is developed in Cartesian coordinates based on the methods of finite sine transform and Hankel transform. On the basis of the analytical solution, the groundwater fluctuations induced by the joint effect of estuarine and oceanic tides are examined. In addition, the influences of physical parameters such as aquitard leakage (L), included angle (Φ), damping coefficient (K_{er}) and separation coefficient (K_{ei}) on the groundwater fluctuations in the confined aquifer are investigated and discussed. The analysed results indicate that those parameters have significant impacts on the head fluctuations in the *L*-shaped coastal leaky aquifer system. The results also show that the effect of L on the normalized amplitude and phase lag of the groundwater fluctuations increases with Φ, but the joint effect of K_{er} and K_{ei} on both parameters decreases as Φ increases.

Key words: analytical model, coastal leaky confined aquifer, groundwater fluctuations, *L*-shaped aquifer

1. INTRODUCTION

The dynamic behavior of groundwater flow in tidal aquifers subject to the tidal effect has received much attention in recent decades. The propagation of the tidal waves along an estuary results in amplitude decay and phase change with distance (Sun, 1997). Tide-induced head fluctuations in a 2-D coastal aquifer system subject to the effects of both oceanic and estuarine tides is therefore complicated. Sun (1997) presented an analytical solution derived from a 2-D transient groundwater flow equation with an estuarine tidal boundary. The predicted head fluctuations in a confined aquifer from his solution showed good match as compared with those of the finite different solutions. Tang and Jiao (2001) displayed a 2-D analytical solution for groundwater head distribution in a coastal leaky confined aquifer. Both the works of Sun (1997) for 2-D groundwater flow in a confined aquifer and Jiao and Tang (1999) for 1-D groundwater flow in a leaky confined aquifer can be considered as special cases of Tang and Jiao (2001). Their solution was also tested against the results obtained from implicit finite difference approximations. Li et al. (2002) developed an analytical model to describe tidal groundwater head fluctuations in an *L*-shaped aquifer, which is bounded by two water-land boundaries with a right angle. The solution of the model is derived using the methods of complex transform and the Green's function.

To our knowledge, the existing solutions for *L*-shaped aquifers are only applicable for the case that the river is perpendicular to the ocean. Yet, the intersection for a river entering the sea in a natural coastline may form an acute or obtuse angle. The objective of this paper is to develop a 2-D analytical model for describing groundwater head fluctuations in a coastal leaky aquifer system with any degree of included angle between the river and the ocean. The solution of the model is developed based on the methods of finite sine transform and Hankel transform along with the transformation between the polar coordinates and Cartesian coordinates. This solution can be used to assess the effects of the oceanic tides and estuarine tides on the groundwater head fluctuations. In addition, the solution can also be used to explore the impacts of leakage and included angle on the head fluctuations and investigate the dynamic groundwater response to various scenarios for the leakage and bending angle of the coastline in tidal leaky confined aquifers.

2. MODEL SETUP AND BOUNDARY CONDITIONS

Consider a coastal leaky aquifer system with an unconfined aquifer on the top, a confined one at the bottom, and an aquitard in between. Figure 1 shows a cross-section view of the conceptual model for an L-shaped aquifer bounded by a river and the ocean with an included angle Φ. The x-axis is horizontal along the seashore while the y-axis is perpendicular to the x-axis and positive toward inland with an angle of $|90 - \Phi|$ between the axis and the river. The joint effect of tidal fluctuations from the ocean and the estuary on the groundwater flow of the confined aquifer is considered. The groundwater fluctuations in the unconfined aquifer is negligible as compared with that in the confined aquifer. These two aquifers, however, interact with each other through the leakage of aquitard. The bottom of the confined aquifer is impermeable. Assume that the aquifer is homogeneous and the thickness of the unconfined aquifer is very large as compared with the magnitude of the tidal fluctuations, therefore allowing linearization of the flow equation for the unconfined aquifer. The flow velocity in the confined aquifer is essentially horizontal, and there is a vertical leakage through the aquitard.

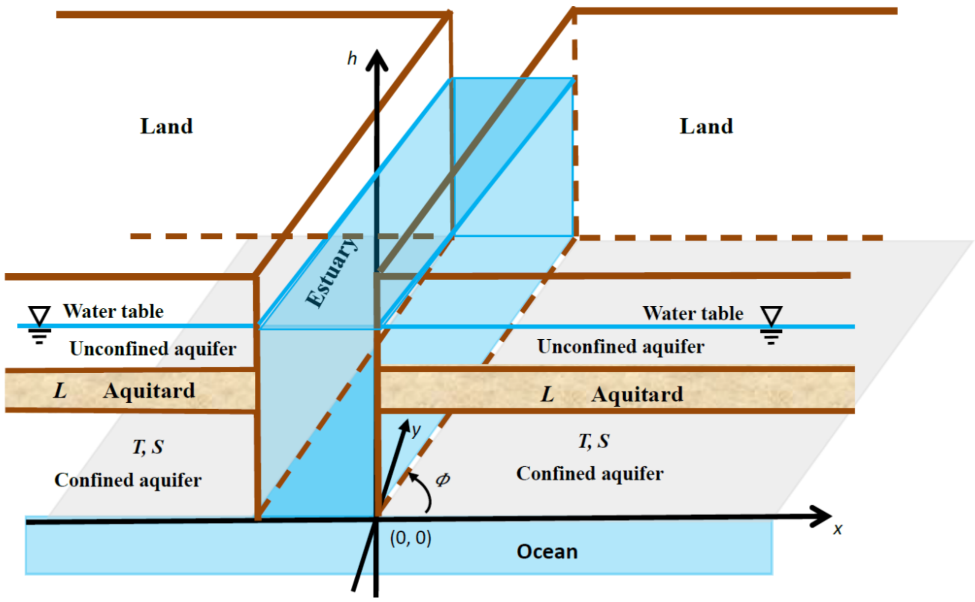

Figure 1. A cross-section view of two-dimensional conceptual model in a leaky anisotropic aquifer system with an arbitrary included angle (Φ) in Cartesian coordinates (x, y).

In addition, the aquitard storage is assumed negligible and the leakage is linearly proportional to the difference in heads of the unconfined and confined aquifers (Li and Jiao, 2001). The initial groundwater head at time t = 0 in the entire system is uniform and equals h_m, which is a distance from the groundwater head to a convenient reference. When t > 0, h_m in the unconfined aquifer remains constant all the times (Jiao and Tang, 1999). Under these assumptions the governing equation for describing the head fluctuations in the confined aquifer ($x > y\cot\phi, y > 0$) can be written as:

$$T\left(\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}\right) = S\frac{\partial h}{\partial t}+L(h-h_m) \tag{1}$$

where $h(x, y, t)$ is the hydraulic head; T and S are the transmissivity and storativity, respectively. The leakage L is defined as the ratio of the hydraulic conductivity of the aquitard to its thickness.

The tidal boundary condition for the ocean is written as

$$h(x,0,t) = h_m + A\cos(\omega t + c) \tag{2a}$$

where $h(x,0,t)$ is the hydraulic head at $y = 0$, i.e., along the seashore, A is the amplitude of the tidal fluctuations, ω is the tidal speed, c is the phase shift, and h_m is the hydraulic head from the datum and commonly assumed to be zero for convenience in the development of analytical solution. Also, $\omega = 2\pi/ t_0$ where t_0 is the tidal period. The estuarine tidal boundary condition is written as

$$h(y\cot\phi, y, t) = h_m + Ae^{-K_{er} y\csc\phi}\cos(\omega t + K_{ei} y\csc\phi + c) \tag{2b}$$

where $h(y\cot\phi, y, t)$ is the hydraulic head along the river bank. The K_{ei} is a separation coefficient defined as the change of phase with distance and K_{er} is a damping coefficient of tidal amplitude (Sun, 1997).

The remote boundary conditions for Eq. (1) at the inland sides in the x- and y-directions are expressed, respectively, as

$$h(x \to \infty, y, t) = 0 \tag{2c}$$

and

$$h(x, y \to \infty, t) = 0 \tag{2d}$$

3. ANALYTICAL SOLUTION

The method of separation of variables is employed first to develop the solution of the model consisted of Eqs. (1) and (2). Let $\overline{h}(x,y)$ be a complex function of the real variables x and y, and assume as

$$h(x, y, t) = \overline{h}(x,y) A \operatorname{Re}(e^{i\omega t + c}) \tag{3}$$

where $i = \sqrt{-1}$ and Re denotes as the real part of the complex expression.

Substituting Eq. (3) into Eq. (1) and dividing the results by $Ae^{i\omega t + c}$ yields

$$\frac{\partial^2 \overline{h}}{\partial x^2} + \frac{\partial^2 \overline{h}}{\partial y^2} - (\overline{L} + i\omega\overline{S})\overline{h} = 0 \tag{4}$$

where $\bar{L} = L/T$ and $\bar{S} = S/T$. The tidal boundary conditions of Eqs. (2a) and (2b) can therefore be expressed, respectively, as

$$\overline{h}(x,0) = 1 \tag{5a}$$

and

$$\overline{h}(y\cot\phi, y) = \operatorname{Re}(e^{-K_e y\csc\phi}) \tag{5b}$$

where $K_e = K_{er} + iK_{ei}$ and the remote boundary conditions of Eqs. (2c) and (2d) respectively become

$$\overline{h}(\infty, y) = 0 \tag{5c}$$

and

$$\overline{h}(x, \infty) = 0 \tag{5d}$$

By changing the Cartesian coordinates (x, y) to the polar coordinates (r, θ), Eq. (4) can be written as

$$\frac{\partial^2 H}{\partial r^2} + \frac{1}{r}\frac{\partial H}{\partial r} + \frac{1}{r^2}\frac{\partial^2 H}{\partial \theta^2} = S_L H \tag{6}$$

where $r = \sqrt{x^2 + y^2}$, $\theta = tan^{-1}(y/x)$, $S_L = \overline{L} + i\omega\overline{S}$, and $H(r,\theta)$ is the hydraulic head expressed in the polar coordinates. Likewise, the tidal boundary conditions, Eqs. (5a) and (5b), can be respectively transformed to the polar coordinates as

$$H(r,0) = 1 \tag{7a}$$

and

$$H(r,\phi) = \mathrm{Re}(e^{-K_e \cdot r}) \tag{7b}$$

and the transformed remote boundary conditions are

$$H(\infty,\theta) = 0 \tag{7c}$$

and

$$H(0,\theta) = 1 \tag{7d}$$

The solution for Eq. (6) subject to the boundary conditions, Eq. (7), developed in the polar coordinates is presented as follows.

Applying the finite sine transform to Eqs. (6) and (7) results in (Yeh and Chang, 2006)

$$\frac{\partial^2 \overline{H}}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{H}}{\partial r} - \frac{\mu_n^2}{r^2}\overline{H} + [1-(-1)^n e^{-K_e \cdot r}]\frac{\mu_n}{r^2} = S_L \overline{H} \tag{8}$$

Then the Hankel transform $\widehat{f}(u) = \int_0^\infty rf(\mathrm{r})J_v\,(ur)dr$ is adopted to eliminate the r coordinate in Eq. (8). The result can be obtained as:

$$\widehat{\overline{H}}\,(u,n) = \frac{\mu_n}{u^2+S_L}\int_0^\infty \{\frac{[1-(-1)^n e^{-K_e \cdot r}]}{r}\}J_{\mu_n}\,(ur)dr \tag{9}$$

where $J_{\mu_n}(\cdot)$ is the Bessel function of the first kind of order μ_n and $\mu_n = n\pi/\phi$ with $n = 1, 2, 3, \ldots$.

Taking the inverse Hankel transform $f(\mathrm{r}) = \int_0^\infty u\,\widehat{f}(u)J_v\,(ur)dr$ first and then the inverse finite sine transform to Eq. (9), the solution of hydraulic head in the polar coordinates can be obtained as:

$$H(r,\theta)=\frac{2}{\phi}\sum_{n=1}^{\infty}\sin(\mu_n\theta)\int_0^{\infty}u_p[1-(-1)^n u_k]\ \mathrm{d}u \tag{10a}$$

with

$$u_p=\frac{uJ_{\mu_n}(\mathrm{ur})}{u^2+S_L} \tag{10b}$$

$$u_k=\frac{(\sqrt{u^2-K_e^2}-K_e)^{\mu_n}}{u^{\mu_n}} \tag{10c}$$

By changing the polar coordinates to the Cartesian coordinates, Eq. (10) can then be transformed as

$$\bar{h}(x,y)=\frac{2}{\phi}\sum_{n=1}^{\infty}\sin[\mu_n tan^{-1}(y/x)]\int_0^{\infty}u_c[1-(-1)^n u_k]\ \mathrm{d}u \tag{11}$$

Substituting Eq. (11) into Eq. (3), the solution for the hydraulic head $h(x, y, t)$ in the confined aquifer written in the Cartesian coordinates can then be denoted as

$$h(x,\ y,\ t)=A\mathrm{Re}\{\frac{2e^{i\omega t+c}}{\phi}\sum_{n=1}^{\infty}\sin[\mu_n tan^{-1}(y/x)]\int_0^{\infty}u_c[1-(-1)^n u_k]\ \mathrm{d}u\} \tag{12a}$$

where

$$u_c=\frac{uJ_{\mu_n}(\mathrm{u}\sqrt{x^2+y^2})}{u^2+S_L} \tag{12b}$$

4. RESULTS AND DISCUSSION

In this section several hypothetical cases given in Jiao and Tang (1999) and Li et al. (2002) are adopted to investigate the effects of leakage (L), included angle (Φ), damping coefficient (k_{er}), and separation coefficient (k_{ei}) on the head fluctuations in the leaky confined aquifer. The values of aquifer parameters in the case studies are: L in the range of 0.001 to 0.01 /hour, k_{er} = k_{ei} ranging from 10^{-6} to 10^{-4}, T = 50 m^2/hour, S = 0.0001, $\omega = 0.2618$ rad/hour, and $h_m = 0$. The amplitude of head fluctuations is normalized as h_A = $|h|$ /A and the phase lag is defined as ph = $\tan^{-1}(-1\times \mathrm{Im}(h)/\mathrm{Re}(h))$ in which Im denotes the imaginary part of the complex expression.

The curves of h_A and ph of the head fluctuations versus Φ for various values of L are plotted in Figure 2. This figure displays the joint effect of L and Φ happened at (x, y) = (212.1m, 212.1m) (i.e., (r, θ) = (300 m, 45°)) for T = 50 m^2/hour, S = 0.0001, K_{er} = K_{ei} = 10^{-5} /hour, and L ranging from 0.001 to 0.01 /hour. The figure indicates that h_A decreases significantly but ph increases slightly when increasing Φ from 50 ° to 120 ° and L < 0.01/hour. The figure, on the other hand, indicates that both h_A and ph decrease apparently as L increases. Obviously, both L and Φ have significant impact on the groundwater fluctuations in the leaky confined aquifer.

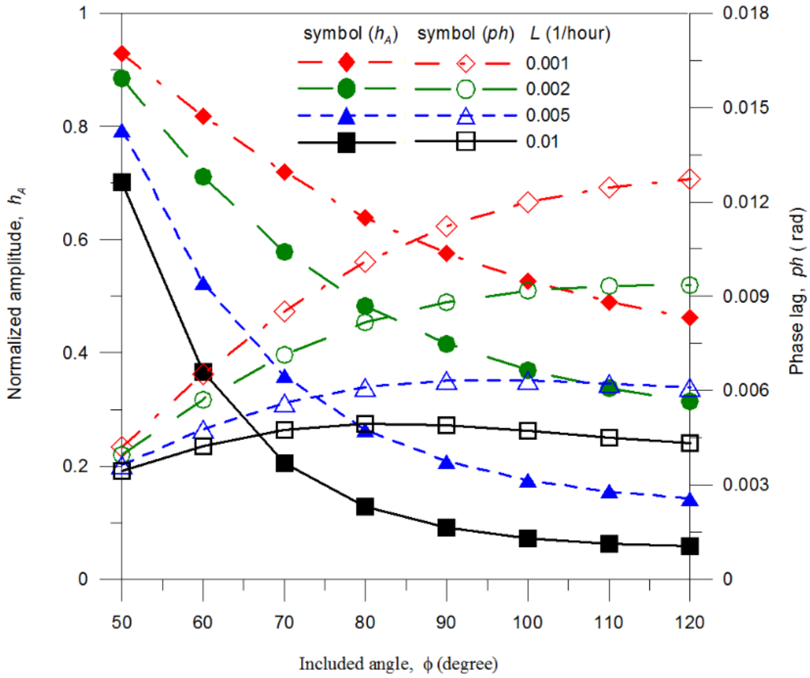

Figure 2. The curves of h_A and ph of the head fluctuations versus Φ happened at θ = 45^o for T = 50 m^2/hour, S = 0.0001, with various leakages at r = 300m

The curves of h_A and ph of head fluctuations versus Φ for various values of k_{er} and k_{ei} are plotted in Figure 3. This figure displays the joint effects of Φ, k_{er}, and k_{ei} occurred at (r, θ) = (3000 m, 45^o) for T = 50 m^2/hour and S = 0.0001 with various values of k_{er} and k_{ei} . The figure shows that h_A decreases markedly but ph increases significantly as Φ increases from 50 ^{o} to 120 ^{o} for L = 0. This figure, however, indicates that h_A decreases but ph increases as k_{er} (= k_{ei}) increases from 10^{-6} to 10^{-4} for Φ in the range between 50 ^{o} and 120 ^{o}. As indicated in the figure, the k_{er} and k_{ei} also have significant influences on the groundwater fluctuations in the estuarine leaky confined aquifer. Figures 2 and 3 displays the effect of L on the h_A and ph of the groundwater fluctuations increases as Φ increases. In contrast, the effect of K_{er} and K_{ei} decreases with increasing Φ.

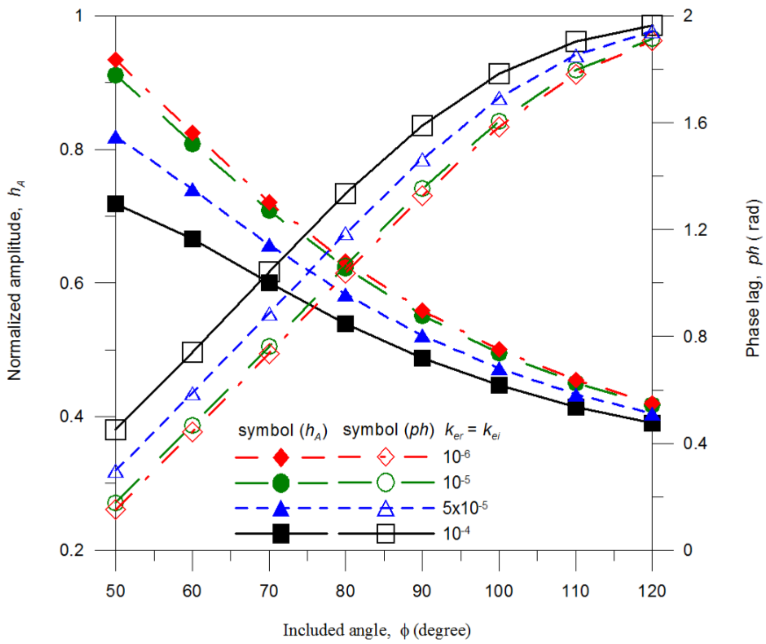

Figure 3. The curves of h_A and ph of the head fluctuations versus Φ happened at θ = 45^o for T = 50 m^2/hour, S = 0.0001, with various values of K_{er} and K_{ei} at r = 3000m.

5. CONCLUDING REMARKS

A new two-dimensional analytical model has been presented to describe the groundwater level fluctuations in an *L*-shaped leaky confined aquifer bounded by a river and the ocean with an arbitrary included angle. The solution of the model expressed in Cartesian coordinates is developed by applying first the finite sine transform and then the Hankel transform. This solution is used to explore the influences of the leakage, included angle, damping coefficient, and separation coefficient on the tidal responses in *L*-shaped leaky aquifer systems. The results display that the normalized amplitude of the groundwater fluctuations decreases markedly but the phase lag of the groundwater fluctuations increases slightly as the included angle increases. On the other hand, both the normalized amplitude and phase lag decrease apparently as the leakage increases. In addition, the normalized amplitude decreases but phase lag increases as the damping and separation coefficients increase. Obviously, these four parameters including the leakage, included angle, damping coefficient, and separation coefficient all have significant impacts on the groundwater level fluctuations in a tidal leaky confined aquifer system.

ACKNOWLEDGMENTS

This study was partly supported by the Taiwan Ministry of Science and Technology under the grants MOST 104-2621-M-130 -003 and 105-2221-E-009 -043-MY2.

REFERENCES

Jiao, J. J., Tang, Z. H., 1999. An analytical solution of groundwater response to tidal fluctuation in a leaky confined aquifer, Water Resour. Res. 35(3): 747-751.

Li, H., Jiao, J. J., 2001. Analytical studies of groundwater-head fluctuation in a coastal confined aquifer overlain by a semi-permeable layer with storage, Adv. Water Resour. 24: 565-573.

Li, H., Jiao, J. J., Luk, M., Cheung, K., 2002. Tide-induced groundwater level fluctuation in coastal aquifers bounded by L-shaped coastlines, Water Resour. Res. 38(3): 1024.

Sun, H., 1997. A two-dimensional analytical solution of groundwater response to tidal loading in an estuary, Water Resour. Res. 33(6): 1429-1435.

Tang, Z., Jiao, J. J., 2001. A two-dimensional analytical solution for groundwater flow in a leaky confined aquifer system near open tidal water, Hydrol. Process. 15(4): 573-585.

Yeh, H. D., Chang, Y. C., 2006. New analytical solutions for groundwater flow in wedge-shaped aquifers with various topographic boundary conditions, Adv. Water Resour. 29(3): 471-480.